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Report on the PhD dissertation of Omar Mohsen

It is a pleasure to report on the thesis of Mr Omar Mohsen, which has been submitted by him for the PhD degree. Mr Mohsen's thesis is an exceptionally fine piece of work that makes amply clear his creativity, skill and judgment. It is worthy of the PhD degree by any international standard. Moreover each of the three separate parts in the thesis stands by itself and merits publication in a leading mathematics journal. It is evident from the thesis that Mr Mohsen has a promising, and quite possibly spectacular, academic career ahead of him.

I will write briefly about each of the chapters in the thesis. But first let me compliment Mr Mohsen on the fine introduction to the thesis (which is not numbered as a chapter in my version). His gift for mathematical exposition is on show throughout the thesis, but it is particularly well displayed in the introduction, which presents very clear and concise summaries of the three works in the thesis.

Chapter One of the thesis is a miniature course on the some of the primary constructions in Alain Connes' noncommutative geometry, focusing on those topics relevant to the two works presented in Chapters Two and Three, especially the deformation to the normal cone construction, and associated groupoids and C*-algebras. This material has not been conveniently organized anywhere else in the mathematical literature that I am aware of, and Mr Mohsen's exposition here is first-rate. (The necessary background for the final work is provided just as clearly in an appendix.)

Turning to the substance of the thesis, the first work, in Chapter Two, concerns Witten's famous approach to the Morse inequalities through the analysis of deformed "Witten

Laplacians" on closed manifolds. Witten's work from well over 30 years ago has inspired numerous other studies, but the development presented here may be the clearest and most compelling of them all.

The essence of Witten's idea is use a Morse function to adjust the de Rham operator by a parameter, *t*, times an order zero term, so that as *t* tends to infinity the adjusted, or deformed, operator, "localizes" near the critical points of the Morse function. This relates the number of critical points, and their types, to the kernel of the de Rham operator, or in other words to cohomology, and such a relation is what the Morse inequalities are all about. The key observation in Chapter Two is that although the deformed operators do not converge to a limit in the ordinary sense, as *t* tends to infinity, they *do* converge to a natural limit on the special fiber, when they are viewed as operators on the standard fibers of the deformation to the normal cone associated to the inclusion of the critical set into the manifold. Moreover one obtains a smooth, regular family of operators this way, with compact resolvent. The Morse inequalities follow nearly immediately.

This is a beautifully simple approach that eliminates, through the use of the appropriate geometrical context, nearly all of the intricate analysis needed elsewhere. It should be noted that Witten's construction of his Morse complex (which strengthens the Morse inequalities) is not considered in the thesis. But it is natural to suppose that the deformation to the normal cone should play a useful and clarifying role in this problem, too.

Chapter Three is about an index theorem that was proved by Erik van Erp in his own thesis, and subsequently published in the Annals of Mathematics. The problem was to compute the Fredholm index of certain subelliptic operators on contact manifolds. It was studied by the geometric analysts (the Melrose school) in some detail, but ultimately without success. The missing idea, which van Erp supplied, was the correct definition of the symbol class in K-theory. This involved noncommutative C*-algebras in a crucial way, and so went a beyond the usual framework of traditional geometric analysis.

With the symbol class in hand, a natural version of Connes' tangent groupoid presented itself (the same groupoid was also studied by Raphael Ponge) and the index problem was solved by van Erp using this groupoid. More accurately, the subelliptic index problem was reduced, using the groupoid together with a deformation argument, to the classical index problem studied by Atiyah and Singer.

The actual construction of the groupoid by van Erp (or Ponge) was a bit involved. Mohsen develops in place of that construction an extremely simple approach using iterated deformations to the normal cone. It is clearly the right way to go. It shortens previous arguments, it generalizes effortlessly, and moreover, as a bonus, it simultaneously creates the geometric context for the deformation-reduction to the Atiyah-Singer problem. The full index theorem is not considered in the thesis, but again, as with the Morse complex, it seems certain to me that the geometric context that Mohsen has created is the right context for the theorem in full.

The final topic in the thesis, presented in Chapter Four, is a bit different in that deformations to the normal cone do not play a central role. Instead, KK-theory is central, and in particular KK-theory with real coefficients as recently studied by Georges Skandalis (Mohsen's advisor) and collaborators. The main result is the construction of a "universal Atiyah-Patodi-Singer alpha-class" in equivariant KK-theory that pulls back to the K-theory classes constructed in the famous work of Atiyah, Patodi and Singer. The construction solves a puzzle that had been outstanding for many years; decades, in fact. And it puts on clear display, one more time, both the power and the clarity of Mr Mohsen's ideas.

In summary, this is an outstanding thesis, full of many distinct accomplishments, rich with possibilities for future work, and beautifully written. I give it my fullest praise.

Yours faithfully,

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