On maximal hypoellipticity and sub-riemannian geometry

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Habilitation à diriger des recherches

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On maximal hypoellipticity and sub-Riemannian geometry

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Abstract

We give an overview of our work on maximally hypoelliptic differential operators and sub-Riemannian geometry. We then discuss open problems and directions in the field.

Résumé

Nous présentons un aperçu de notre travail sur les opérateurs différentiels hypoelliptiques maximaux et la géométrie sous-riemannienne. Nous discutons ensuite des problèmes ouverts dans le domaine.

Contents

Introduction (en Français)

Dans cette thèse, nous présentons un aperçu de notre travail sur les opérateurs différentiels hypoelliptiques maximaux et la géométrie sous-riemannienne. Les origines de cette dernière remontent aux travaux de Chow [Cho39], qui a démontré qu'étant donné certains champs de vecteurs $X_1, \cdots, X_n$ sur une variété lisse M, il est possible de relier deux points quelconques via des chemins dont la dérivée, en chaque point, appartient à l'espace linéaire engendré par $X_1, \cdots, X_n$, à condition que ces champs de vecteurs satisfassent la condition dite de Crochet de Lie de Hörmander.

En 1967, Hörmander [Hör67] a établi un lien entre la géométrie sous-riemannienne et l'analyse, en prouvant son célèbre théorème de la somme des carrés, qui affirme que si $D = X_1^2 + \cdots + X_n^2$ et que $v \in C^\infty(M)$ est une fonction lisse, alors les solutions de l'équation $Du = v$ sont également lisses. Il est important de noter que cette propriété de régularité des solutions est également satisfaite par le Laplacien en géométrie riemannienne. Le Laplacien est un exemple d'opérateur différentiel elliptique. Les opérateurs différentiels elliptiques constituent une large classe d'opérateurs possédant des propriétés de régularité très fortes (voir Théorème 1.4). Cependant, la somme des carrés de

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Hörmander $X_1^2 + \cdots + X_n^2$ n'est, en général, pas un opérateur elliptique. Motivés par le résultat de Hörmander, Folland, Helffer, Nourrigat, Rothschild, et Stein [FS74; RS76; HN85] ont défini la classe des opérateurs différentiels hypoelliptiques maximaux, qui peuvent être considérés comme l'analogue des opérateurs elliptiques en géométrie sous-riemannienne.

Ces dernières années, nos travaux se sont concentrés sur l'étude des opérateurs différentiels hypoelliptiques maximaux en utilisant des méthodes issues de la géométrie non commutative.

En 1982, Connes a introduit la C^*-algèbre d'un feuilletage régulier [Con82], grâce à laquelle il a généralisé le théorème de l'indice d'Atiyah-Singer pour les feuilletages. Ce travail a marqué le début du domaine de la géométrie non commutative, qui utilise les outils des algèbres d'opérateurs pour résoudre des problèmes en géométrie et en analyse. S'appuyant sur une construction de Connes [Con94], appelée groupoïde tangent, Debord et Skandalis [DS14] ont proposé une nouvelle définition géométrique des opérateurs pseudo-différentiels classiques. Ils ont également démontré que cette construction pouvait être utilisée pour prouver des propriétés de régularité des solutions d'opérateurs différentiels elliptiques.

L'un de nos principaux résultats et outils est une généralisation du groupoïde tangent à la géométrie sous-riemannienne. L'idée de généraliser la construction de Connes remonte aux travaux de thèse de van Erp [Erp10a] et Ponge [Pon08], qui ont proposé une telle construction dans le cas des variétés de contact (voir Exemple 2.1.2).

Nos travaux récents portent sur l'application de cette généralisation du groupoïde tangent de Connes [Moh24d] pour l'étude des opérateurs différentiels hypoelliptiques maximaux, ainsi que pour la généralisation du théorème de l'indice d'Atiyah-Singer et le calcul des espaces tangents en géométrie sous-riemannienne, au sens de Gromov. Plus précisément, cette thèse est divisée en six chapitres, qui sont organisés comme suit :

1. Dans section 1, nous donnons une brève introduction aux opérateurs différentiels elliptiques. Nous énonçons également le principal théorème de régularité pour les opérateurs différentiels elliptiques.

2. Dans section 2, nous introduisons la classe des opérateurs différentiels maximaux hypoelliptiques. Nous énonçons notre théorème, une généralisation du théorème de régularité pour les opérateurs différentiels elliptiques aux opérateurs maximaux hypoelliptiques.

3. Dans section 3, nous donnons une brève introduction à la géométrie sous-riemannienne. Dans [Gro96], Gromov a définit l'espace tangent d'un espace métrique quelconque. Il est naturel de se demander ce qu'est l'espace tangent en géométrie sous-riemannienne. Dans [Bel96], Bellaiche calcule l'espace tangent en géométrie sous-riemannienne. Dans cette section, nous énonçons notre deuxième théorème principal qui montre que le calcul de Bellaiche n'est pas complet. Nous calculons ensuite tous les espaces tangents en géométrie sous-riemannienne.

4. Section 4 est consacré au théorème de l'indice d'Atiyah-Singer. Nous énonçons notre troisième théorème principal qui est une formule topologique (généralisant celle d'Atiyah-Singer) pour l'indice analytique des opérateurs différentiels maximaux hypoelliptiques sur les variétés compactes.

5. La démonstration des trois résultats précédents repose sur la même construction/technique, celle de groupoïde tangent. Dans section 5, nous présentons cette construction ainsi que notre généralisation et sa connexion avec les trois théorèmes précédents.

1 Regularity theorem for elliptic operators

Let M be a smooth manifold and $D : C^\infty(M) \to C^\infty(M)$ a differential operator. We denote by $C^{-\infty}(M)$ the space of distributions on M, i.e., the topological dual of $C^\infty_c(M, |\Lambda|^1 TM)$. The differential operator D naturally extends to a linear map $C^{-\infty}(M) \to C^{-\infty}(M)$. Furthermore, the extension is pseudo-local, i.e.,

$$\mathrm{singsupp}(Du) \subseteq \mathrm{singsupp}(u), \quad \forall u \in C^{-\infty}(M)$$

We are interested in the linear partial differential equation

$$Du = v, \quad u, v \in C^{-\infty}(M)$$

More precisely, we are interested in the question: if v is smooth, is u smooth? Motivated by this question, L. Schwartz introduced the following :

Definition 1.1. We say that D is hypoelliptic if for any $u \in C^{-\infty}(M)$, $\mathrm{singsupp}(Du) = \mathrm{singsupp}(u)$. Equivalently, if $D(u)$ is smooth on an open set $U \subseteq M$, then u is also smooth on U.

Examples 1.2.
1. The differential operators $\frac{d}{dx}$ on $\mathbb{R}$ is hypoelliptic. This is just the fundamental theorem of calculus.
2. The Cauchy-Riemann equation $\frac{\partial}{\partial y} - i\frac{\partial}{\partial x}$ on $\mathbb{R}^2$ is also hypoelliptic.
3. Trivially, the differential operator $\frac{\partial}{\partial x}$ on $\mathbb{R}^2$ is not hypoelliptic.

It is no coincidence that we give the Cauchy-Riemann equation as an example. In fact some properties of analytic functions are shared with solutions of more general hypoelliptic differential operators. We share here two such folklore properties.

Proposition 1.3. *Let D be a hypoelliptic differential operator on a smooth manifold M.*

1. *For any $n \in \mathbb{N}$, $K \subseteq M$ compact, $K' \subseteq M$ a compact neighborhood of K, there exists $C > 0$ such that if $f \in \ker(D)$, then $\|f\|_{C^n(K)} \leqslant C \|f\|_{C^0(K')}$.*
2. *If M is compact, then $\ker(D)$ is finite dimensional.*

Proof. We prove the first statement under the hypothesis M is compact and $K = K' = M$. We leave the general case to the reader. We equip $C^\infty(M)$ with a locally convex topology τ given by the semi-norms $f \mapsto \|f\|_{C^0(M)}$ and $f \mapsto \|D(f)\|_{C^n(M)}$ for all $n \in \mathbb{N}$. This topology is complete because if $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, then there exists a $f \in C^0(M)$ and $g \in C^\infty(M)$ such that $f_n \to f$ in $C^0(M)$ and $D(f_n) \to g$ in the classical $C^\infty(M)$ topology. It follows that $D(f) = g$ in the sense of distributions. By hypoellipticity, we deduce that $f \in C^\infty(M)$ from which completeness follows. By the open mapping theorem applied to $\mathrm{Id} : C^\infty(M) \to C^\infty(M)$, where on the codomain we put the τ-topology and on the domain we put the classical topology, we deduce that τ and the classical topology coincide. Hence, for any $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$, $C > 0$ such that

$$\|f\|_{C^n(M)} \leqslant C(\|f\|_{C^0(M)} + \|D(f)\|_{C^m(M)}), \quad \forall f \in C^\infty(M).$$

By restricting to $\ker(D)$, the result follows.

The first result together with Arzela-Ascoli theorem imply that $\{f \in \ker(D) : \|f\|_{C^0(M)} \leqslant 1\}$ is compact. By Riesz theorem, the second result follows. ☐

The general method to prove that a differential operator is hypoelliptic is by constructing a parametrix, i.e., a linear operator $P : C^{-\infty}(M) \to C^{-\infty}(M)$ such that

1. P is pseudo-local. In particular $P(C^{\infty}(M)) \subseteq C^{\infty}(M)$.

2. $R = PD - \mathrm{Id}$ is regularizing, i.e., $R(C^{-\infty}(M)) \subseteq C^{\infty}(M)$.

This method in particularly successful for elliptic operators. To define elliptic operators, we recall the definition of the classical principal symbol of a differential operator D. In local coordinates, D can be written

$$D(f) = \sum_{|I| \leqslant n} g_I \frac{\partial^{|I|}}{\partial x_I}(f)$$

We define the principal symbol by

$$\sigma(D, x, \xi) = \sum_{|I|=n} g_I(x) \sqrt{-1}^n \xi_I,$$

where if $I = (i_1, \cdots, i_n)$, then $\xi_I = \xi_{i_1} \cdots \xi_{i_n}$. One checks that the principal symbol locally defined above gives a well-defined coordinate independent smooth function $\sigma(D) : T^*M \to \mathbb{C}$.

Theorem 1.4 (Main regularity theorem for elliptic operators, see [Hör85])**.** *Let D be a differential operator of order n on a smooth manifold M. The following are equivalent*

1. *For every $(x, \xi) \in T^*M \backslash 0$, $\sigma(D, x, \xi) \neq 0$.*

2. *For any $s \in \mathbb{R}$, $u \in C^{-\infty}(M)$, $Du \in H^s(M)$ implies that $u \in H^{s+n}(M)$.*

3. *For any differential operator D' of order $\leqslant n$, and any compact $K \subseteq M$, there exists $C > 0$ such that*

$$\left\|D'(f)\right\|_{L^2(K)} \leqslant C(\|D(f)\|_{L^2(K)} + \|f\|_{L^2(K)}), \quad \forall f \in C_c^{\infty}(K)$$

Furthermore if M is compact, then the above is equivalent to

4. *For any $s \in \mathbb{R}$, the operator $D : H^{s+n}(M) \to H^s(M)$ is Fredholm.*

If D satisfies the above conditions, then D is called elliptic.

By Sobolev lemma, $C^{\infty}(M) = \bigcap_{s \in \mathbb{R}} H^s(M)$, it follows that elliptic differential operators are hypoelliptic.

2 Maximally hypoelliptic differential operators

Let M be a smooth manifold, $X_1, \cdots, X_n$ be vector fields on M. We say that $X_1, \cdots, X_n$ satisfy Hörmander's condition if for any $x \in M$, T_xM is linearly spanned by $X_1(x), \cdots, X_n(x)$, $[X_i, X_j](x), [[X_i, X_j], X_k](x), \cdots$. We will suppose that the number of commutators needed to generate T_xM is bounded above by some natural number $N \in \mathbb{N}$ independent of x.

Examples 2.1.

1. For any $k \in \mathbb{N}$, the vector fields $\frac{\partial}{\partial x}$ and $x^k \frac{\partial}{\partial y}$ on $\mathbb{R}^2$ satisfy Hörmander's condition.

2. A contact structure on a smooth manifold is essentially the same as some vector fields $X_1, \cdots, X_n$ which satisfy the conditions $\mathrm{rank}(X_1(x), \cdots, X_n(x)) = \dim(M) - 1$ and T_xM is linearly spanned by $X_1(x), \cdots, X_n(x)$, $[X_i, X_j](x)$ for all $x \in M$.

Notice that in Examples 2.1, Hörmander's condition is satisfied in a rather non-continuous way. By this we mean that around $x \neq 0$, one doesn't need any commutators to generate T_xM but near $x = 0$, one needs k-iterated commutators to generate T_xM. Studying this behavior is at the center of our work.

The interest in Hörmander's condition comes from the following celebrated result by Hörmander [Hör67].

Theorem 2.2 (Hörmander)**.** *The differential operator $\Delta_X = X_1^2 + \cdots + X_n^2$ is hypoelliptic.*

The differential operator Δ_X is elliptic if and only if at every $x \in M$, T_xM is linearly spanned by $X_1(x), \cdots, X_n(x)$. Our main goal is to generalize the notion of elliptic operators to include the operator Δ_X.

The following simple proposition follows directly from Hörmander's condition

Proposition 2.3. *Any differential operator D can be written as $D = P(X_1, \cdots, X_n)$ where P is a noncommutative polynomial with coefficients in $C^\infty(M)$.*

Sketch of proof. A commutator $[X, Y] = XY - YX$ is by definition a noncommutative polynomial in X, Y. Hörmander's condition implies that any vector field X can be written as a polynomial in $X_1, \cdots, X_n$. The result follows. ☐

Thanks to Proposition 2.3, we can make the following definition

Definition 2.4. The Hörmander order of a differential operator D is the minimum degree of P such that $D = P(X_1, \cdots, X_n)$.

Example 2.5. In Examples 2.1.1, the Hörmander order of $\frac{\partial}{\partial y}$ is $k + 1$.

Motivated by Theorem 1.4.3, we introduce the following:

Definition 2.6. A differential operator D is called maximally hypoelliptic if for any differential operator D' whose Hörmander order is less than or equal to the Hörmander order of D, and for any compact $K \subseteq M$, there exists $C > 0$ such that

$$\left\|D'(f)\right\|_{L^2(K)} \leqslant C(\|D(f)\|_{L^2(K)} + \|f\|_{L^2(K)}), \quad \forall f \in C_c^\infty(K). \tag{1}$$

We can also define maximal hypoellipticity using Sobolev spaces as follows: Let $k \in \mathbb{N}$. We define

$$\tilde{H}^k(M) := \{u \in L^2_{\text{loc}}M : D(u) \in L^2_{\text{loc}}M \quad \forall D \text{ whose Hörmander order } \leqslant k\}. \tag{2}$$

One can extend $\tilde{H}^k(M)$ for any $k \in \mathbb{R}$ by duality and interpolation. This is possible because the definition of $\tilde{H}^k(M)$ in (2) satisfies the sheaf condition, i.e., if $u \in L^2_{\text{loc}}M$, such that for every $x \in M$, there exists $\chi \in C_c^\infty(M)$ such that $\chi(x) \neq 0$ and $\chi u \in \tilde{H}^k(M)$, then $u \in \tilde{H}^k(M)$. One can show the following

Proposition 2.7 ([AMY22])**.** *Let D be a differential operator whose Hörmander order is n. The following are equivalent :*

1. *The operator D is maximally hypoelliptic.*
2. *For any $s \in \mathbb{R}$, and $u \in C^{-\infty}(M)$, $Du \in \tilde{H}^s(M)$ implies that $u \in \tilde{H}^{s+n}(M)$.*

Notice that by Proposition 2.3, $\bigcap_{s\in\mathbb{R}} \tilde{H}^s(M) = C^\infty(M)$. In particular, a maximally hypoelliptic differential operator is hypoelliptic. Like in Theorem 1.4, one also wants to define a principal symbol which detects maximal hypoellipticity. Let $\mathfrak{g}$ be the free Lie algebra generated by $\tilde{X}_1, \cdots, \tilde{X}_n$ with the only relation being that any iterated commutator of length $> N$ vanishes. The Lie algebra $\mathfrak{g}$ is nilpotent finite dimensional. Let G be the simply connected Lie group integrating $\mathfrak{g}$. Let $\pi : G \to U(H)$ be a unitary irreducible representation of G,

$$d\pi : \mathfrak{g} \to \mathrm{End}(C^\infty(\pi))$$

the differential of π, where $C^\infty(\pi) \subseteq H$ is the space of smooth vectors. We can now define our principal symbol of D. One writes D as a polynomial $D = P(X_1, \cdots, X_n)$ as in Proposition 2.3 with $\deg(P)$ equal to the Hörmander order of D (i.e., with $\deg(P)$ as minimal as possible). This step is analogous to the step of writing D in local coordinates when defining the classical principal symbol. We then define

$$\sigma(D, x, \pi) = P_{max,x}(d\pi(\tilde{X}_1), \cdots, d\pi(\tilde{X}_n)) \in \mathrm{End}(C^\infty(\pi)),$$

where $P_{\max,x}$ is the polynomial P after removing all lower order terms and replacing each coefficient $f \in C^\infty(M)$ with $f(x)$. One obvious issue with this definition of the principal symbol is that it is not clear if it is well-defined. In other words, if $D = P(X_1, \cdots, X_n) = Q(X_1, \cdots, X_n)$ for some polynomials P, Q, then it is not obvious if $P_{max,x}(d\pi(\tilde{X}_1), \cdots, d\pi(\tilde{X}_n)) = Q_{max,x}(d\pi(\tilde{X}_1), \cdots, d\pi(\tilde{X}_n))$. Well, turns out that the principal symbol is not well-defined. But, it is well-defined for a very special set of representations which we call the Helffer-Nourrigat set $\mathcal{T}_x$. This set of representations will be described in details at the end of Section 3.

Theorem 2.8 ([AMY22])**.** *For any $x \in M$, $\pi \in \mathcal{T}_x$, D differential operator, $\sigma(D, x, \pi)$ is well-defined.*

We can now state the main theorem of this section.

Theorem 2.9 ([AMY22])**.** *Let M be a smooth manifold, $X_1, \cdots, X_n$ be vector fields satisfying Hörmander's condition, D a differential operator of Hörmander order n. The following are equivalent*

1. *The differential operator is maximally hypoelliptic.*
2. *For any $x \in M$, $\pi \in \mathcal{T}_x \backslash \{1_G\}$, $\sigma(D, x, \pi)$ is injective.*

If M is compact, then the above is equivalent to the following

3. *For every $s \in \mathbb{R}$, the operator $D : \tilde{H}^{s+n}(M) \to \tilde{H}^s(M)$ is left-invertible modulo compact operators.*

The main idea behind the proof of Theorem 2.9 will be briefly discussed in Section 5. Theorem 2.9 was conjectured by Helffer and Nourrigat in [HN79b], see also [HN85; HN05]. Notice that if $X_1(x), \cdots, X_n(x)$ linearly span T_xM at every $x \in M$ without any need for commutators, then Theorem 2.9 is precisely Theorem 1.4. Theorem 2.9 was known in the following two cases:

1. In the case where $X_1(x), \cdots, X_n(x), [X_i, X_j](x)$ linearly span T_xM at every $x \in M$ by Helffer and Nourrigat [HN85].
2. In the case where the vector fields $X_1, \cdots, X_n$ are equiregular by Rothschild [Rot79].

   Definition 2.10. We say that the vector fields $X_1, \cdots, X_n$ are equiregular if for any $k \in \mathbb{N}$, the rank of the linear subspace of T_xM linearly spanned by $X_1, \ldots, X_n(x)$ and all iterated commutators of length $\leqslant k$ is locally constant (as a function in x).

This is the case for contact manifolds for example.

In addition to the above, many authors have worked on Theorem 2.9, either restricting to the case where the manifold is a simply connected nilpotent Lie group, or to a specific differential operator, or building a pseudo-differential calculus which contains parametrix for maximally hypoelliptic differential operators in various degrees of generalities, see [Bou74; FS74; Dyn76; Goo76; RS76; Bea77; Dyn78; Roc78; HN79a; Mel82; Mel83; Tay84; BG88; Cum89; EMM91; Gło91; Chr+92; Pon08; BFG09; FR14; Str14; FR16; EY19; FF20; Ewe21; DH22].

The main advance in our work is studying the non equiregular case (for instance Examples 2.1.1). As we will see in Section 3 this condition has various strong implications on the sub-Riemannian geometry associated to $X_1, \cdots, X_n$.

So far, we have only used L^2-norms. A theorem due to Street shows that one can also use L^p-norms if $p \in]1, +\infty[$.

Theorem 2.11 (Street [Str14])**.** *Let D be a maximally hypoelliptic differential operator on a smooth manifold M. Then for any differential operator D' whose Hörmander order is less than or equal to the Hörmander order of D, $p \in]1, +\infty[$, and for any compact $K \subseteq M$, there exists $C > 0$ such that*

$$\left\|D'(f)\right\|_{L^p(K)} \leqslant C(\|D(f)\|_{L^p(K)} + \|f\|_{L^p(K)}), \quad \forall f \in C_c^\infty(K).$$

We now return to Theorem 2.2. Hörmander in fact proved the following as well :

Theorem 2.12 (Hörmander)**.** *Given vector fields $X_1, \cdots, X_n$ satisfying Hörmander's condition The differential operator $\Delta'_X = X_1^2 + \cdots + X_{n-1}^2 + X_n$ is hypoelliptic.*

The differential operator Δ'_X is not maximally hypoelliptic in the traditional sense defined above. We now generalise the notion of maximal hypoellipticity to include the above example as well. Suppose we are given some natural numbers $v_1, \cdots, v_n \in \mathbb{N}$. We can then define the weighted Hörmander order of any differential operator D to be the minimum of the weighted degree of P such that $D = P(X_1, \cdots, X_n)$, where the weighted degree of $P(x_1, \cdots, x_n)$ is calculated by giving x_i the weight v_i. Everything discussed above now extends with appropriate modifications in this weighted setting. The main remarks we wish to highlight are the following:

- The definition of Sobolev spaces (2) is only valid if $k \in \mathbb{N}$ is divisible by $v_1, \cdots, v_n$. For other values of k, one has to define Sobolev spaces by interpolation and duality.
- One defines maximally hypoellipticity using Sobolev spaces (if $u \in C^{-\infty}(M)$, $Du \in \tilde{H}^s(M)$, then $u \in \tilde{H}^{s+n}(M)$ where n is the weighted Hörmander order of D) instead of Definition 2.6. This condition is equivalent to (1) as long as one supposes that the Hörmander order of D is divisible by $v_1, \cdots, v_n$.
- Theorem 2.9 and Theorem 2.11 hold with no assumptions on weights (other than $v_i \in \mathbb{N}$ for all i) and no assumptions on the Hörmander order of D.
- The operator Δ'_X is now an example of a maximally hypoelliptic differential operator with weights $v_1 = \cdots = v_{n-1} = 1$ and $v_n = 2$.

All the results of Section 4 and Section 5 are also true if one adds weights $v_1, \cdots, v_n \in \mathbb{N}$ to $X_1, \cdots, X_n$ in full generality with no further assumptions on weights (other than $v_i \in \mathbb{N}$ for all i). To simplify discussion, we will ignore weights for the rest of this text.

3 Sub-Riemannian geometry

Let M be a smooth manifold, $X_1, \cdots, X_n$ be vector fields satisfying Hörmander's condition, $x, y \in M$. A sub-Riemannian path from x to y is an absolutely continuous path $\gamma : [0,1] \to M$ such that $\gamma(0) = x$, $\gamma(1) = y$, and $\gamma'(t) \in \operatorname{span}(X_1(\gamma(t)), \cdots, X_n(\gamma(t)))$ almost everywhere in t.

Theorem 3.1 ([Cho39])**.** *Let $x, y \in M$. There exists a sub-Riemannian path from x to y*

Given $x \in M$ and $v \in \operatorname{span}(X_1(x), \cdots, X_n(x)) \subseteq T_xM$, we define

$$\|v\| := \inf \left\{ \sqrt{\sum_{i=1}^{n} v_i^2} : v = \sum_{i=1}^{n} v_i X_i(x) \right\}.$$

If γ is a sub-Riemannian path from x to y, we define

$$l(\gamma) = \int_0^1 \|\dot{\gamma}(t)\| \, dt$$

We can then define the sub-Riemannian distance (also called the Carnot-Carathéodory distance)

$$d_{CC}(x, y) = \inf\{l(\gamma)\}$$

Proposition 3.2. *The distance d_{CC} generates the usual topology on M.*

Following Gromov [Gro96], if (X, d) is a metric space, then the limit $\lim_{t\to 0^+}(X, t^{-1}d, x)$, if it exists, in the sense of pointed Gromov-Hausdorff distance is called the tangent cone of X at x. The tangent cone of a Riemannian manifold (M, d_{Riem}) at x is $(T_xM, d_{\text{Riem}}, 0)$. We are interested in computing the tangent cone of (M, d_{CC}).

Generalizing a result of Mitchell [Mit85], Bellaïche [Bel96] computed the tangent cone of (M, d_{CC}) as follows: Let G be the simply connected nilpotent Lie group from Section 2. For each $x \in M$, Bellaïche identifies a simply connected Lie subgroup $\mathfrak{r}_x \subseteq G$ of codimension $\dim(M)$. He shows that

$$\lim_{t\to 0^+}(M, t^{-1}d_{CC}, x) = (G/\mathfrak{r}_x, d_{G/\mathfrak{r}_x}, \mathfrak{r}_x) \tag{3}$$

where $d_{G/\mathfrak{r}_x}$ is a Carnot-Carathéodory metric on the homogeneous space $G/\mathfrak{r}_x$.

The main observation of our work [Moh24d] is that the limit in (3) is not uniform in x. That is, if $x_n \in M$ is a sequence converging to x and $t_n \to 0^+$, then $\lim_{n\to+\infty}(M, t_n^{-1}d_{CC}, x_n)$ exists (up to taking a subsequence) but the limit is of the form $(G/H, d_{G/H}, H)$ where $H \subseteq G$ is a simply connected Lie group and in general $H \neq \mathfrak{r}_y$ for all $y \in M$.

This leads to our construction. For each $x \in M$, we define a subset $\mathcal{G}_x^0$ of simply connected Lie subgroups of G of codimension $\dim(M)$, which satisfies the following conditions

1. $\mathfrak{r}_x \in \mathcal{G}_x^0$.
2. If $g \in G$, $H \in \mathcal{G}_x^0$, then $gHg^{-1} \in \mathcal{G}_x^0$.
3. The set $\mathcal{G}_x^0$ is closed when identified (using Lie algebras) with a subset of the Grassmannian manifold of subspaces of $\mathfrak{g}$ of codimension $\dim(M)$.

The set $\mathcal{G}_x^0$ comes from looking at linear relations between the vector fields $X_1, \cdots, X_d$ and their commutators which we now define. Let $\mathfrak{g}$ be the Lie algebra of G. There is a natural linear map

$$\natural : \mathfrak{g} \to \mathcal{X}(M),$$

where $\mathcal{X}(M)$ is the space of vector fields on M. The map $\natural$ is the unique linear map which sends $\tilde{X}_i$ to X_i and which sends iterated Lie brackets of $\tilde{X}_i$ in $\mathfrak{g}$ of length $\leqslant N$ to the corresponding Lie brackets in $\mathcal{X}(M)$. Notice that $\natural$ isn't a Lie algebra homomorphism because iterated Lie brackets in $\mathcal{X}(M)$ of length $> N$ do not necessarily vanish. For $x \in M$, let $\natural_x : \mathfrak{g} \to T_xM$ be the composition of $\natural$ with the evaluation map at $x \in M$. The map $\natural_x$ is surjective by Hörmander's condition. Therefore, $\ker(\natural_x)$ is a subspace of $\mathfrak{g}$ of codimension equal to $\dim(M)$. Let $\mathrm{Grass}(\mathfrak{g})$ be the Grassmannian manifold of subspaces of codimension $\dim(M)$. We also make use of the natural graded dilation $\alpha_t : \mathfrak{g} \to \mathfrak{g}$ on $\mathfrak{g}$ which is defined by $\alpha_t(\tilde{X}_i) = t\tilde{X}_i$, $\alpha_t([\tilde{X}_i, \tilde{X}_j]) = t^2[\tilde{X}_i, \tilde{X}_j]$, etc. The main point of our work [Moh24d] is that the convergence in the Gromov-Hausdorff distance of $(M, t_n^{-1}d_{CC}, x_n)$ can be reduced to the linear algebra problem of computing the limit of $\alpha_{t_n^{-1}}(\ker(\natural_{x_n}))$ in the Grassmannian manifold $\mathrm{Grass}(\mathfrak{g})$. We thus define

$$\mathcal{G}_x^0 := \{\mathfrak{h} \in \mathrm{Grass}(\mathfrak{g}) : \exists (t_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+^\times, (x_n)_{n\in\mathbb{N}} \subseteq M \text{ such that } t_n \to 0, x_n \to x, \alpha_{t_n^{-1}}(\ker(\natural_{x_n})) \to \mathfrak{h}\}.$$

Even though $\natural$ is not a Lie algebra homomorphism and therefore $\ker(\natural_x)$ is not a Lie subalgebra of $\mathfrak{g}$ in general. We prove that the limit $\alpha_{t_n^{-1}}(\ker(\natural_{x_n}))$, if it exists, is always a Lie subalgebra of $\mathfrak{g}$. Therefore, $\mathcal{G}_x^0$ can be considered as a set of Lie subgroups of G.

We go further by creating a space that includes all the spaces $\mathcal{G}_x^0$ together as x varies. To this end, we consider the inclusion

$$\begin{aligned} M \times \mathbb{R}_+^\times &\to \mathrm{Grass}(\mathfrak{g}) \times M \times \mathbb{R}_+ \\ (x,t) &\mapsto (\alpha_{\frac{1}{t}}(\ker(\natural_x)), x, t) \end{aligned}$$

We take the space

$$\mathbb{G}^0 := M \times \mathbb{R}_+^\times \bigsqcup_{x\in M} \mathcal{G}_x^0 \times \{(x,0)\}$$

to be the closure of $M \times \mathbb{R}_+^\times$ inside $\mathrm{Grass}(\mathfrak{g}) \times M \times \mathbb{R}_+$ equipped with the subspace topology. This topology is a second countable locally compact metrizable topology which makes the natural projection $\mathbb{G}^0 \to M \times \mathbb{R}$ a proper continuous map. Our main theorem is the following

Theorem 3.3 ([Moh24d]). *If $(x_n, t_n) \in M \times \mathbb{R}_+^\times$ converges in $\mathbb{G}^0$ to $(H, x, 0)$ where $H \in \mathcal{G}_x^0$, i.e., if $t_n \to 0$, $x_n \to x$ and $\alpha_{t_n^{-1}}(\ker(\natural_{x_n})) \to \mathfrak{h}$, then*

$$\lim_{n\to+\infty} (M, t_n^{-1}d_{CC}, x_n) = (G/H, d_{G/H}, H),$$

where $d_{G/H}$ is a Carnot-Carathéodory metric on the homogeneous space G/H.

Since the projection map $\mathbb{G}^0 \to M \times \mathbb{R}$ is proper, it follows that Theorem 3.3 gives all possible tangent cones. The main idea behind the proof of Theorem 3.3 turns out to be the same as that of Theorem 2.9 which is discussed in Section 5.

We remark that if the vector fields $X_1, \cdots, X_n$ are equiregular, then $\mathcal{G}_x^0 = \{\mathfrak{r}_x\}$. In other words, the limit in (3) is uniform in x.

We end this section by giving the definition of the set $\mathcal{T}_x$ from Theorem 2.8. There are three equivalent ways to define the set $\mathcal{T}_x$.

1. An irreducible unitary representation π of G belongs to $\mathcal{T}_x$ if and only if π is weakly contained in the induced representation $G \to U(L^2(G/H))$ for some $H \in \mathcal{G}_x^0$.

2. If we identify $\mathcal{T}_x$ with a subset of $\mathfrak{g}^*$ using Kirillov's orbit method [Kir62; Bro73], then

$$\mathcal{T}_x = \bigcup_{H \in \mathcal{G}_x^0} \mathfrak{h}^\perp$$

where $\mathfrak{h}$ is the Lie algebra of H. Here, $\mathfrak{h}^\perp = \{\xi \in \mathfrak{g}^* : \xi(\mathfrak{h}) = 0\}$.

3. If we identify $\mathcal{T}_x$ with a subset of $\mathfrak{g}^*$ using Kirillov's orbit method, then

$$\mathcal{T}_x = \{\xi \in \mathfrak{g}^* : \exists (t_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+^\times, (x_n, \xi_n)_{n\in\mathbb{N}} \subseteq T^*M \ni t_n \to 0, x_n \to x, \xi_n \circ \natural_{x_n} \circ \alpha_{t_n} \to \xi\} \tag{4}$$

All these various ways to define $\mathcal{T}_x$ together with Theorem 2.9 suggest that the set $\mathcal{T}_x$ should be seen as the cotangent space in sub-Riemannian geometry. We remark that the set $\mathcal{T}_x$ was introduced by Helffer and Nourrigat in [HN79b] using (4).

4 Index theorem

In [AS68], Atiyah and Singer found a topological formula for the analytic index of elliptic differential operators on compact manifolds. Their formula depends on the classical principal symbol. Given the analogies between Theorem 1.4 and Theorem 2.9, it is natural to wonder if one can also formulate an index theorem for maximally hypoelliptic differential operators using our principal symbol. In this section, we will formulate such an index theorem based on our work [Moh22c].

Let M be a compact smooth manifold and $X_1, \cdots, X_n$ be vector fields satisfying Hörmander's condition.

Proposition 4.1 ([AMY22]). *Let D be a differential operator on M of Hörmander order n. The following are equivalent :*

1. *The operator D and its formal adjoint D^* (with respect to a Riemannian metric) are maximally hypoelliptic.*
2. *For any $x \in M$, $\pi \in \mathcal{T}_x \backslash \{1_G\}$, $\sigma(D, x, \pi) : C^\infty(\pi) \to C^\infty(\pi)$ is bijective.*
3. *For every $s \in \mathbb{R}$, the operator $D : \tilde{H}^{s+n}(M) \to \tilde{H}^s(M)$ is Fredholm.*

We say that D is bi-maximally hypoelliptic if it satisfies the above conditions.

We remark that if D_1 and D_2 are the formal adjoints of D with respect to two different Riemannian metrics, then $D_1 - D_2$ has strictly less Hörmander order than D_1 and D_2, so D_1 is maximally hypoelliptic if and only if D_2 is maximally hypoelliptic.

For the following proposition, if $T : V \to W$ is a linear map between vector spaces, then we say that T has finite analytic index if $\dim(\ker(T))$ and $\operatorname{codim}(\operatorname{im}(T))$ are finite. In this case,

$$\operatorname{ind}_a(T) := \dim(\ker(T)) - \operatorname{codim}(\operatorname{im}(T))$$

Proposition 4.2 ([Moh22c]). *Let D be a bi-maximally hypoelliptic differential operator. The following linear maps have finite analytic index. Furthermore, their analytic indices coincide.*

$$D : C^\infty(M) \to C^\infty(M)$$
$$D : \tilde{H}^{s+n}(M) \to \tilde{H}^s(M), \quad \forall s \in \mathbb{R}$$
$$D : C^{-\infty}(M) \to C^{-\infty}(M)$$

To formulate our index theorem, we need to define a C^*-algebra denoted by A. Consider $C^*(G) \otimes C(M)$ where G is the simply connected nilpotent Lie group from Section 2. The spectrum (the space of irreducible unitary representations) of $C^*(G) \otimes C(M)$ is $\hat{G} \times M$ where $\hat{G}$ is the space of irreducible unitary representations of G. The following

$$\bigsqcup_{x \in M} \mathcal{T}_x \times \{x\} \subseteq \hat{G} \times M$$

is a closed subset of the spectrum. By general theory of C^*-algebras (see [Dix77]), a closed subset of the spectrum corresponds to a quotient C^*-algebra of $C^*(G) \otimes C(M)$ which we denote by A.

Let D be a bi-maximally hypoelliptic differential operator. By construction, the spectrum of A is $\bigsqcup_{x \in M} \mathcal{T}_x \times \{x\}$. For each such representation, we have an invertible operator $\sigma(D, x, \pi) : C^\infty(\pi) \to C^\infty(\pi)$. One can put all these operators together to obtain an element $[\sigma(D)] \in K_0(A)$, the K-theory of the C^*-algebra A.

We now construct a linear map $\mathrm{ind}_t : K_0(A) \to \mathbb{Z}$ which we call the topological index. We first need to compute $K_0(A)$. Let

$$\mathcal{T} := \bigsqcup_{x \in M} \mathcal{T}_x \times \{x\}$$

seen as a subset of $\mathfrak{g}^* \times M$ using Kirillov's obit method. The space $\mathcal{T}$ is a closed subset of $\mathfrak{g}^* \times M$, so it is locally compact. We denote by $K^0(\mathcal{T})$, its topological K-theory with compact support, see [AS68].

Proposition 4.3 ([Moh22c]). *One has a natural isomorphism* $\mathrm{CT} : K_0(A) \to K^0(\mathcal{T})$.

Recall that the Connes-Thom isomorphism [FS80; Con81] implies that

$$K^0(C^*G \otimes C(M)) \simeq K^0(\mathfrak{g}^* \times M).$$

The isomorphism CT can be seen as a localization of the Connes-Thom isomorphism. Naturality in Proposition 4.3 refers to the fact that CT is compatible with Mayer–Vietoris sequence which allows one to make concrete computations.

Now, let us recall that the set $\mathcal{T}_x$ is defined in Equation (4) using a limit procedure from T^*M. One can use Equation (4) to define a topological space

$$Z = T^*M \times \mathbb{R}_+^\times \sqcup \mathcal{T} \times \{0\} \tag{5}$$

More precisely, the topology on Z is determined by the following three properties

1. The set $T^*M \times \mathbb{R}_+^\times$ is open subset with its usual topology.
2. The set $\mathcal{T} \times \{0\}$ is a closed subset with the subspace topology from $\mathfrak{g}^* \times M$.
3. A sequence $(x_n, \xi_n, t_n) \in T^*M \times \mathbb{R}_+^\times$ converges to $(x, \xi, 0)$ if and only if $x_n \to x$, $t_n \to 0$, and $\xi_n \circ \natural_{x_n} \circ \alpha_{t_n} \to \xi$.

One can show that Z is locally compact Hausdorff second countable. Since $T^*M \times \mathbb{R}_+^\times$ is contractible, the topological space Z (think of it as a cone with base $\mathcal{T}$) defines an excision map

$$\mathrm{Ex} : K^0(\mathcal{T}) \to K^0(T^*M).$$

Finally, Atiyah and Singer famously defined a topological index map

$$\mathrm{Ind}_t : K^0(T^*M) \to \mathbb{Z}.$$

We can now state the main theorem of this section

Theorem 4.4 ([Moh22c])**.** *For any bi-maximally hypoelliptic differential operator on a compact smooth manifold M,*

$$\mathrm{Ind}_a(D) = \mathrm{Ind}_t(\mathrm{Ex}(\mathrm{CT}([\sigma(D)]))).$$

Theorem 4.4 provides yet another support for the claim that $\mathcal{T}_x$ should be seen as the cotangent space in sub-Riemannian geometry. Theorem 4.4 generalizes Atiyah-Singer index theorem because for elliptic operators $A = C_0(T^*M)$, and CT and Ex are the identity maps. Theorem 4.4 was known for contact manifolds by the work of van-Erp [Erp10a; Erp10b], see also [BE14]. Our proof (and our index formula) relies heavily on the ideas introduced by van-Erp.

Amusingly, like Theorem 2.9 and Theorem 3.3, the proof of Theorem 4.4 also relies on the construction from Section 5.

5 Blow up spaces

5.1 Connes's tangent groupoid

Let M be a smooth manifold. The deformation to the normal cone (see [Ful84]) of $M \times M$ along the diagonal produces a smooth manifold

$$\mathbb{TM} := M \times M \times \mathbb{R}_+^\times \cup TM \times \{0\}$$

The topology on $\mathbb{T}M$ is characterized by the following three properties

1. The set $M \times M \times \mathbb{R}_+^\times$ is an open subset of $\mathbb{T}M$ with its usual topology.
2. The set TM is a closed subset of $\mathbb{T}M$ with its usual topology.
3. A sequence $(y_n, x_n, t_n) \in M \times M \times \mathbb{R}_+^\times$ converges to $(x, \xi, 0) \in TM$ if and only if $t_n \to 0$, $x_n, y_n \to x$, and in any local coordinates around x, $\frac{y_n - x_n}{t_n} \to \xi$.

In [Con94], Connes made a central remark that $\mathbb{T}M$ is a Lie groupoid. This essentially means that $C_c^\infty(\mathbb{T}M)$ has a convolution product

$$\begin{aligned} C_c^\infty(\mathbb{T}M) \times C_c^\infty(\mathbb{T}M) &\to C_c^\infty(\mathbb{T}M) \\ f * g(z, x, t) &= t^{-\dim(M)} \int_M f(z, y, t) g(y, x, t) \\ f * g(x, \xi, 0) &= \int_{T_x M} f(x, \eta, 0) g(x, \xi - \eta, 0) \end{aligned}$$

Here the integrals are with respect to some Riemannian metric on M. We remark that one can more elegantly replace functions with density valued functions, see [Con94; DL10]. This removes the $t^{-\dim(M)}$ factor. This is intimately connected to the heat kernel asymptotics and Weyl law of elliptic operators.

The groupoid $\mathbb{T}M$ has been studied by many authors. We will list here some of these applications which are relevant to our discussion.

1. In [Con94], Connes showed that the groupoid $\mathbb{T}M$ can give a very quick and elegant proof of Atiyah-Singer index theorem.
2. In [DS14; DS15], Debord and Skandalis showed that the groupoid $\mathbb{T}M$ can be used to prove Theorem 1.4. More precisely they prove the following theorem:

Theorem 5.1 (Debord and Skandalis)**.** *Let $k \in \mathbb{C}$ with $\Re(k) < 0$, $f \in C_c^\infty(\mathbb{T}M)$. The kernel*

$$P(y,x) = \int_0^\infty t^{-\dim(M)} f(y,x,t) \frac{dt}{t^{k+1}} \tag{6}$$

defines a classical pseudo-differential operator on M of order k.

Again, the factor $t^{-\dim(M)}$ disappears if we replace functions with densities. Theorem 5.1 can be extended to arbitrary $k \in \mathbb{C}$, but one requires additional hypotheses on f for the integral (6) to converge in the sense of distributions. For example if $0 \leqslant \Re(k) < 1$, then one needs $\int_{T_xM} f(x,\xi,0)d\xi = 0$ for all $x \in M$. For higher $\Re(k)$, one requires more vanishing conditions. The elegance of Theorem 5.1 is that all the technical conditions one usually requires in the classical approach to pseudo-differential operators (for example [Hör71]) become the very simple condition $f \in C_c^\infty(\mathbb{T}M)$. Using Theorem 5.1, Debord and Skandalis showed that the groupoid $\mathbb{T}M$ can be used to reconstruct the short exact sequence for classical pseudo-differential operators which easily implies Theorem 1.4.

Another approach very similar to the approach of Debord and Skandalis was found by van-Erp and Yuncken [EY19].

3. As we said in Section 3, the tangent space of $(M, d_{\mathrm{Riem}}, x)$ is $(T_xM, d_{\mathrm{Riem}}, 0)$. This result can be seen as a consequence of the fact that the function

$$d : \mathbb{T}M \to \mathbb{R}, \quad (y,x,t) \mapsto \frac{d_{\mathrm{Riem}}(y,x)}{t}, \quad (x,\xi,0) \mapsto d_{\mathrm{Riem}}(\xi,0)$$

is continuous. We remark that at the same time as Connes constructed his tangent groupoid, Pansu [Pan83] calculated the tangent space of nilpotent groups. The main step of his proof is the construction of a deformation space like $\mathbb{T}M$. He then proves that the appropriate distance function on his deformation space is continuous. His main result is then immediately obtained from this continuity.

The above suggests that to prove Theorem 2.9 and Theorem 3.3, one needs to construct the analogue of Connes's tangent groupoid in sub-Riemannian geometry. This is precisely our main contribution to the work described above. We will describe our construction in the next section. We refer the reader to [Moh24d] for a detailed construction of our groupoid.

The idea of constructing an analogue of Connes's tangent groupoid in sub-Riemannian geometry goes back to van-Erp's thesis [Erp10a; Erp10b] and Ponge's thesis [Pon08]. In van-Erp's thesis, he constructed this groupoid for contact manifolds (using Darboux's theorem). He then showed that this groupoid gives an index theorem for maximally hypoelliptic differential operators on contact manifolds. Previous to our construction which holds for any vector fields satisfying Hörmander's condition, multiple authors obtained different (but equivalent) constructions of this groupoid in the equiregular case, see [EY17; HH18; CP19; Moh21].

5.2 Tangent groupoid in sub-Riemannian geometry

The idea behind our construction is very simple: the tangent groupoid is a Riemannian geometry construction in which one adds the tangent spaces at $t = 0$. Therefore, the analogue in sub-Riemannian geometry has to be the space in which one adds the tangent spaces in sub-Riemannian geometry at $t = 0$. Our groupoid as a set is equal to

$$\mathbb{G} = M \times M \times \mathbb{R}_+^\times \sqcup \bigsqcup_{\substack{x \in M \\ H \in \mathcal{G}_x^0}} G/H \times \{0\}.$$

We equip $\mathbb{G}$ with a natural topology. We will only describe the most interesting feature of $\mathbb{G}$ which is how a sequence $(y_n, x_n, t_n) \in M \times M \times \mathbb{R}_+^\times$ converges to a point $(gH, x, 0)$ where $x \in M$, $H \in \mathcal{G}_x^0$, $gH \in G/H$.

Proposition 5.2. *A sequence (y_n, x_n, t_n) converges to $(gH, x, 0)$ if and only if the following conditions are satisfied:*

1. *$x_n, y_n \to x$ and $t_n \to 0$*
2. *The sequence $(x_n, t_n, 0)$ converges to $(H, x, 0)$ in the topology of $\mathbb{G}^0$*
3. *There exists $v_n \in G$ such that $\exp(\alpha_{t_n}(v_n)) \cdot x_n = y_n$, and $v_n \to v$ for some $v \in gH$.*

Here $\exp(\alpha_{t_n}(v_n)) \cdot x_n = y_n$, means that the flow of the vector field $\natural(\alpha_{t_n}(v_n))$ starting from x_n arrives at y_n at time 1. We remark here to define $\natural : G \to \mathcal{X}(M)$, we identify G with $\mathfrak{g}$ using the exponential map.

Notice the analogy between $\exp(\alpha_{t_n}(v_n)) \cdot x_n = y_n$ and $v_n \to v \in gH$ and the identity $\frac{y_n - x_n}{t_n} \to \xi$ which appears in the definition of Connes's tangent groupoid. We can now state our main theorem on $\mathbb{G}$.

Theorem 5.3 ([Moh24d])**.** *The space $\mathbb{G}$ is locally compact Hausdorff second countable space, which is furthermore a topological groupoid whose space of objects is $\mathbb{G}^0$.*

In the previous theorem that the space $\mathbb{G}$ being Hausdorff is essentially a consequence of the period bounding lemma [AR67; Ozo72]. The use of the period bounding lemma to construct topological groupoids goes back to the PhD thesis of Debord, see [Deb01; Deb13].

The space $\mathbb{G}$ is unfortunately not a smooth manifold in general. In general, one can construct a closed topological embedding $\mathbb{G} \to \mathbb{R}^n$ for n big enough. One can then define smooth functions on $\mathbb{G}$ to be the restriction of smooth functions on $\mathbb{R}^n$. More natural definitions of $C^\infty(\mathbb{G})$ which don't depend on such embeddings can also be given, see [Moh24c].

The fact that $\mathbb{G}$ is not a smooth manifold introduces all sort of technical difficulties, especially in adapting the work of Debord and Skandalis, and van-Erp and Yuncken [DS14; DS15; EY19] to the maximal hypoellipticity setting. An important tool which is used here is the work of Androulidakis and Skandalis [AS09] where they introduce the notion of bi-submersions which serve as local charts for $\mathbb{G}$.

We remark that $\mathbb{G}$ is a special case of a much more general construction which we gave in [Moh22a]. We end by mentioning the recent work of Louis [Lou24], where he gave variants and generalizations of our construction from [Moh22a]. He also studied the question of when $\mathbb{G}$ (and his more general spaces) are smooth manifolds.

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