On maximal hypoellipticity and sub-riemannian geometry

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Structure of talk

1. Connes's proof of the Atiyah-Singer index theorem.
2. A construction by van Erp.
3. Our generalisation of van Erp's construction.
4. 3 different applications of our construction.

History

- In 1957, Grothendieck proved the Grothendieck-Riemann-Roch theorem. His proof uses K-theory and the deformation to the normal cone construction.
- In 1963, Atiyah-Singer proved the Atiyah-Singer index theorem using cobordism theory. In 1968, their first article "On the index of elliptic operators: I" where they used instead K-theory.

> The reader who is familiar with the Riemann-Roch theorem will realize that our original proof of the index theorem was modelled closely on Hirzebruch's proof of the Riemann-Roch theorem. Naturally enough we were led to look for a proof modelled more on that of Grothendieck. While we have not completely succeeded in this aim, we have at least found a proof which is much more natural, does not use cobordism, and lends itself therefore to generalization. In spirit, at least, it has much in common with Grothendieck's approach.

- In 1990, Connes published his book "Noncommutative Geometry" where he gave a new proof of the Atiyah-Singer index theorem.

Deformation to the normal cone construction

Let M be a smooth manifold, $V \subseteq M$ a smooth submanifold, N the normal bundle of V in M. The space

$$\mathrm{DNC}(M, V) := M \times \mathbb{R}_+^* \sqcup N \times \{0\}$$

has a natural smooth structure where the following functions are smooth:

1. The projection

$$\mathrm{DNC}(M, V) \to M \times \mathbb{R}_+, \quad (x, t) \mapsto (x, t), \quad (n, x, 0) \mapsto (x, 0).$$

2. $$\mathrm{DNC}(M, V) \to \mathbb{R}, \quad (x, t) \mapsto \frac{f(x)}{t}, \quad (n, x, 0) \mapsto df_x(n),$$

   where $f : M \to \mathbb{R}$ is smooth which vanishes on V.

Connes's proof of Atiyah-Singer

Connes considers

$$\mathbb{T}M = M \times M \times \mathbb{R}_+^* \sqcup TM \times \{0\}.$$

Let us describe the topology in more details.

- The set $M \times M \times \mathbb{R}_+^*$ is open with its usual topology
- The set $TM \times \{0\}$ is closed with its usual topology
- A sequence (y_n, x_n, t_n) converges to $(X, x, 0)$ if and only if
  1. $$y_n, x_n \to x, \text{ and } t_n \to 0$$
  2. $$\frac{y_n - x_n}{t_n} \to X$$

     in a local chart around x.

Connes's proof of Atiyah-Singer

If $f, g \in C_c^\infty(\mathbb{T}M)$, then we define $f * g \in C_c^\infty(\mathbb{T}M)$

$$f * g(z, x, t) = t^{-\dim(M)} \int_M f(z, y, t) f(y, x, t) dy$$

$$f * g(X, x, 0) = \int_{T_x M} f(X - Y, x, 0) g(Y, x, 0) dY$$

Proposition (Connes)

The convolution of two smooth functions is smooth. So, $C_c^\infty(\mathbb{T}M)$ is an algebra

In fact it is a $*$-algebra.

$$f^*(y, x, t) = \bar{f}(x, y, t), \quad f^*(X, x, 0) = \bar{f}(-X, x, 0).$$

Connes's proof of Atiyah-Singer

One can complete the $*$-algebra $C_c^\infty(\mathbb{T}M)$ to a C^*-algebra A.

$$\|f\| = \sup\left\{ \left\| t^{-\dim(M)} f(\cdot,\cdot,t) \right\|_{B(L^2(M))} : t \in \mathbb{R}_+^* \right\}$$

The C^*-algebra A is a C^*-algebra fibered over $\mathbb{R}_+$ whose fiber at $t \neq 0$ is $K(L^2M)$ and at $t = 0$ is $C_0(T^*M)$.

Connes's argument : Replace K-theory of topological spaces by K-theory of C^*-algebras in Grothendieck's proof.

Generalisations

Many generalisations of Connes's construction have been found by different authors. We are interested in a generalisation by van Erp in his thesis in 2005.

van Erp's construction

$$\mathbb{T}M = M \times M \times \mathbb{R}_+^* \sqcup TM \times \{0\}.$$

- The set $M \times M \times \mathbb{R}_+^*$ is open with its usual topology
- The set $TM \times \{0\}$ is closed with its usual topology
- A sequence (y_n, x_n, t_n) converges to $(X, x, 0)$ if and only if
  1. $$y_n, x_n \to x, \text{ and } t_n \to 0$$
  2. $$\frac{y_n - x_n}{t_n} \to X$$

  in a local chart around x.

van Erp's construction

Let M be a smooth manifold, $H \subseteq TM$ a subbundle of corank 1. For any $x \in M$,

$$\begin{aligned} \omega : & H_x \times H_x \to T_xM/H_x \\ & (a, b) \mapsto [X, Y](x) \mod H_x \end{aligned}$$

where $X, Y \in \Gamma(H)$, $X(x) = a$ and $Y(x) = b$.
Exercise : prove that it doesn't depend on choice of X, Y.
The bundle H is contact if and only if ω is nondegenerate.
A group law on $H_x \oplus T_xM/H_x$:

$$(a, n) \cdot (b, m) = (a + b, n + m + \frac{\omega(a, b)}{2}).$$

So, $H \oplus TM/H$ is a bundle of Heisenberg groups.

van Erp's construction

Let (M, H) be a contact manifold. Consider

$$\mathbb{T}_H M = M \times M \times \mathbb{R}_+^* \sqcup H \oplus TM/H \times \{0\}.$$

The space $\mathbb{T}_H M$ is a smooth manifold.

- The set $M \times M \times \mathbb{R}_+^*$ is open with its usual topology
- The set $H \oplus TM/H \times \{0\}$ is closed with its usual topology
- A sequence (y_n, x_n, t_n) converges to $(X, x, 0)$ if and only if
  1. $y_n, x_n \to x$, and $t_n \to 0$.
  2. $$\frac{y_n - x_n}{t_n} \to X$$
     in a local chart around x.
  3. $$\frac{y_n - x_n}{t_n} \to X$$
     in a local Darboux chart around x.
  4. $$\frac{y_n x_n^{-1}}{t_n} \to X$$

van Erp's construction

If $f, g \in C_c^\infty(\mathbb{T}_H M)$, then we define $f * g \in C_c^\infty(\mathbb{T}_H M)$

$$f * g(z,x,t) = t^{-(\dim(M)+1)} \int_M f(z,y,t)f(y,x,t)dy$$

$$f * g(X,x,0) = \int_{H_x M \oplus T_x M/H_x} f(XY^{-1},x,0)g(Y,x,0)dY$$

We also have adjoint

$$f^*(y,x,t) = \bar{f}(x,y,t), \quad f^*(X,x,0) = \bar{f}(X^{-1},x,0).$$

Proposition (van Erp)

The convolution and adjoint of smooth functions is smooth. So, $C_c^\infty(\mathbb{T}_H M)$ is a $*$-algebra

The $*$-algebra $C_c^\infty(\mathbb{T}_H M)$ can be completed to a C^*-algebra.

Tangent groupoid in sub-Riemannian geometry

Let $X_1, \cdots, X_n$ be vector fields which satisfy Hörmander's condition, for any $x \in M$,

$$X_1(x), \cdots, X_n(x), [X_i, X_j](x), [[X_i, X_j], X_k](x), \cdots$$

span T_xM. For simplicity, we suppose the number of commutators at any $x \in M$ needed is bounded above by N.

Examples

1. For any $k \in \mathbb{N}$, the vector fields $\frac{\partial}{\partial x}$ and $x^k \frac{\partial}{\partial y}$ on $\mathbb{R}^2$ satisfy Hörmander's condition.
2. If (M, H) is a contact manifold, then take $X_1, \cdots, X_n \in \Gamma(H)$ be generators.

Tangent groupoid in sub-Riemannian geometry

- Let $\mathfrak{g}$ be the free nilpotent Lie algebra with n generators $\tilde{X}_1, \cdots, \tilde{X}_n$ and depth N.
- Let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$.
- Linear map

$$\natural : \mathfrak{g} \to \mathcal{X}(M), \quad \natural(\tilde{X}_i) = X_i, \natural([\tilde{X}_i, \tilde{X}_j]) = [X_i, X_j], \cdots$$

  The map $\natural$ is not a Lie algebra homomorphism.
- If $X \in \mathfrak{g}$, then $\exp(X) \cdot x$ is the time 1-flow of $\natural(X)$ starting at x.
- The vector space $\mathfrak{g}$ is naturally graded, so we Lie algebra homomorphisms

$$\alpha_t : \mathfrak{g} \to \mathfrak{g}, \quad \alpha_t(\tilde{X}_i) = t\tilde{X}_i.$$

Starting point of our construction is: Start with $y, x \in M$, and look at

$$\alpha_{t^{-1}}(\{X \in \mathfrak{g} : y = \exp(X) \cdot x\})$$

Tangent groupoid in sub-Riemannian geometry

$$\mathbb{G} = M \times M \times \mathbb{R}_+^* \sqcup \sqcup_{x \in M} \sqcup_{H \in \mathcal{G}_x^0} G/H \times \{0\}.$$

1. The set $M \times M \times \mathbb{R}_+^*$ is open with its usual topology
2. A sequence (y_n, x_n, t_n) converges to $(A, x, 0)$ if and only if
    1. $y_n, x_n \to x$, and $t_n \to 0$.
    2. $$\alpha_{t_n^{-1}}(\{X \in \mathfrak{g} : y_n = \exp(X) \cdot x_n\}) \to A$$

Tangent groupoid in sub-Riemannian geometry

Let

$$\natural_x : \mathfrak{g} \to T_xM, \quad \natural_x = \mathrm{ev}_x \circ \natural$$

Hörmander condition = the map $\natural_x$ is surjective.
The kernel of $\natural_x$ is the tangent space at 0 of

$$\{X \in \mathfrak{g} : \exp(X) \cdot x = x\}$$

$H \in \mathcal{G}_x^0$ if and only if there exists $x_n \to x$ and $t_n \to 0$ such that

$$\mathfrak{h} = \lim_{n \to +\infty} \alpha_{t_n^{-1}}\left(\ker(\natural_{x_n})\right)$$

Tangent groupoid in sub-Riemannian geometry

The condition

$$\alpha_{t_n^{-1}}(\{X \in \mathfrak{g} : y_n = \exp(X) \cdot x_n\}) \to gH$$

is equivalent to

1. $\alpha_{t_n^{-1}}(\ker(\natural_{x_n})) \to \mathfrak{h}$
2. There exists $v_n \in \alpha_{t_n^{-1}}(\{X \in \mathfrak{g} : y_n = \exp(X) \cdot x_n\})$ such that $v_n \to v \in gH$.

Tangent groupoid in sub-Riemannian geometry

The space $\mathbb{G}$ is a Hausdorff locally compact second countable space.
Hausdorff follows from

Theorem (Period bounding lemma)

Let M be a smooth manifold, $X \in \mathcal{X}(M)$ a vector field, $K \subseteq M$ a compact set. There exists $\epsilon > 0$ such that for any $x \in K$, either $X(x) = 0$ or $\exp(tX) \cdot x \neq x$ for all $0 < t < \epsilon$.

Proof.

Exercise : Use Rolle's theorem and compactness of the spheres. □

Tangent groupoid in sub-Riemannian geometry

The space $\mathbb{G}$ is not a smooth manifold, but one can still define $C_c^\infty(\mathbb{G})$. Let $f, g \in C_c^\infty(\mathbb{G})$

$$f * g(z, x, t) = W(z, x, t) \int_M f(z, y, t) f(y, x, t) dy$$

$$f * g(A, x, 0) = \int_{BC=A} f(B, x, 0) g(C, x, 0) dB$$

One can also define adjoint

Proposition

The convolution and adjoint of two smooth functions is smooth. So, $C_c^\infty(\mathbb{G})$ is a $*$-algebra

One can complete $C_c^\infty(\mathbb{G})$ to a C^*-algebra.

Applications

First application: Criteria for maximal hypoellipticity.

Elliptic operators

Theorem (Kohn, Nirenberg, Hörmander, $\cdots$)

Let D be of order k. Then TFAE

1. $\forall (x, \xi) \in T^*M \backslash 0$, $\sigma(x, \xi) \neq 0$.
2. $\forall s \in \mathbb{N}$, *and any distribution* u, $Du \in H^s(M)$ *implies* $u \in H^{s+k}(M)$.

Moreover, if M is compact then the above is equivalent to

4. *for any* $k \in \mathbb{N}$, $D : H^{s+k}(M) \to H^s(M)$ *is Fredholm.*

Connection with Connes's tangent groupoid

To prove the previous theorem, one uses pseudo-differential operators.

Definition

A pseudo-differential operator is a Schwartz kernel $P(y,x)$ on $M \times M$ such that

1. P is smooth off the diagonal.
2. Near the diagonal, P has the form

$$P(y,x) = \int_{\mathbb{R}^n} e^{i\langle \xi, y-x\rangle} a(x,\xi) d\xi,$$

   where a is a smooth function which admits an asymptotic expansion in homogeneous functions.

Connection with Connes's tangent groupoid

Theorem (Debord and Skandalis)

Let $k \in \mathbb{C}$ with $\Re(k) < 0$, $f \in C_c^\infty(\mathbb{T}M)$. The kernel

$$P(y,x) = \int_0^\infty t^{-\dim(M)} f(y,x,t) \frac{dt}{t^{k+1}}$$

defines a classical pseudo-differential operator on M of order k. Furthermore, all pseudo-differential operators of order k are of this form.

- Invariance under change of coordinates is immediate
- Convolution of smooth functions is smooth implies that convolution of pseudo-differential operators is a pseudo-differential operator.
- The principal symbol of P is given by

$$\sigma(P,x,\xi) = \int_0^\infty \hat{f}(t\xi, x, 0) \frac{dt}{t^{1+k}}$$

- The theorem can be extended to $k \in \mathbb{C}$.

First application

Replace $\mathbb{T}M$ with our space $\mathbb{G}$

Theorem (Androulidakis, M., Yuncken 2022)

Let $X_1, \cdots, X_n$ be vector fields satisfying Hörmander's condition on a smooth manifold M, D be of Hörmander order k. Then TFAE

1. *D is maximally hypoelliptic.*
2. *for any $x \in M$, $\pi \in \mathcal{T}_x^* \subseteq \hat{G} \backslash \{1_G\}$ (set of irreducible unitary representations), $\tilde{\sigma}(D, x, \pi)$ is injective.*

Moreover, if M is compact then the above is equivalent to

3. *for any $s \in \mathbb{N}$, $D : \tilde{H}^{k+s}(M) \to \tilde{H}^s(M)$ is left invertible modulo compact operators.*

Many people have worked on this, see habilitation thesis.

Advantage of this approach

Proposition

If A is a C^*-algebra fibered over $\mathbb{R}_+$, then for any $a \in A$,

$$\limsup_{t \to 0^+} \|a_t\| \leq \|a_0\|$$

Apply the above to the completion of $C_c^\infty(\mathbb{G})$, then use Cotlar-Stein lemma to transform the above identity to identity on operator norm of pseudo-differential operators.

Second application: Index theory.

Index theory

Let $X_1, \cdots, X_n$ be vector fields satisfying Hormander's condition on a compact manifold

Definition

A $*$-maximally hypoelliptic differential operator is a differential operator D such that

$$\pi(D) : C^\infty(\pi) \to C^\infty(\pi)$$

is bijective for each nontrivial representation in the Helffer-Nourrigat set.

Corollary

If D is $*$-maximally hypoelliptic, then $D : C^\infty(M) \to C^\infty(M)$ has finite dimensional kernel and cokernel.

So,

$$\mathrm{Ind}_a(D) = \dim(\ker(D)) - \mathrm{codim}(\mathrm{im}(D)) < +\infty$$

Index theorem

Theorem (M. 2022)

Let $X_1, \cdots, X_m$ be vector fields satisfying Hörmander's condition, D $*$-maximally hypoelliptic on any compact manifold M. Then

$$\mathrm{Ind}_a(D) = \mathrm{Ind}_t(\sigma(D))$$

Special cases (contact manifolds) obtained by van Erp and Baum.

Example

Let $k \in \mathbb{N}$ be arbitrary. Consider

$$\partial_x, \quad (1 - \cos(x))^k \partial_y$$

on $\mathbb{T}^2 = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$. Let $g \in C^\infty(\mathbb{T}^2)$.

$$D = \partial_x^{4k} - (1 - \cos(x))^{2k} \partial_y^2 + ig(x, y)\partial_y$$

D is $*$-maximally hypoelliptic iff

$$g(0, y) \notin \pm\mathrm{spec}(\text{Some Schrodinger operator}), \quad \forall y \in S^1$$

For any λ in the spectrum, let $w(g(0, y), \lambda)$ be winding number. Then

$$\mathrm{Ind}_a(D) = \quad \textstyle\sum_\lambda w(g(0, y), \lambda) - w(g(0, y), -\lambda)$$

Applications

Third application: Computation of tangent space in sub-riemannian geometry.

Gromov metric convergence

Gromov has a notion of convergence of a sequence of pointed metric spaces (X_n, d_n, x_n) to a metric space (X, d, x).

Example

If d is a Riemannian metric on M, then

$$\lim_{t \to 0^+} (M, \frac{d}{t}, x) = (T_x M, d_{T_x M}, 0).$$

Sub-Riemannian metric

Theorem (Chow–Rashevskii 1938)

Let $x, y \in M$. Then there exists a path $\gamma : [0,1] \to M$ connecting x to y such that

$$\gamma'(t) \in \mathrm{span}(X_1(\gamma(t)), \cdots, X_m(\gamma(t))) \quad \forall t.$$

$$d_{CC}(x, y) := \inf \mathrm{length}(\gamma).$$

Theorem (M. 2022)

If $x_n \to x$ and $t_n \to 0^+$, $\alpha_{t_n^{-1}}(\ker(\natural_{x_n})) \to \mathfrak{h}$, then

$$\lim_{n\to\infty}(M, \frac{d_{CC}}{t_n}, x_n) = (G/H, d_{CC}, H).$$

Thank you for your attention